The Importance Of Modeling Correlation
A Loan Guarantee Problem

Gary Schurman, MBE, CFA

April 2012

You are a Credit Analyst at a bank that is considering making ABC Company a $1,000,000 loan at 10% simple interest due and payable in one year. ABC Company will use the loan proceeds to purchase computer equipment. The loan is secured by the computer equipment and guaranteed by ABC's parent XYZ Company. Should ABC default on the loan XYZ will be required to step in and payoff the loan per the terms of the guarantee. If the financial condition of XYZ is such that it cannot make good on the guarantee then the computer equipment securing the loan will be sold with the proceeds from sale going to pay down the loan. Should XYZ default on the guarantee it is anticipated that there will be no recovery over and above the salvage value of the equipment.

For your credit analysis you make the following estimates:

1 Given the financial condition of ABC Company the one year probability that ABC will default on the loan is 0.20.
2 Given the financial condition of XYZ Company the probability that XYZ will not have the financial wherewithal to cover the guarantee should ABC default on the loan is 0.10.
3 Given the business lines of ABC Company and XYZ Company the default correlation between the two companies is 0.60.
4 The salvage value of the computer equipment at the end of year one is $300,000.

The Problem: What is the bank's total expected return on the proposed transaction? The bank requires a minimum expected rate of return of 5.00%. Is the interest rate on the loan high enough to support the bank's minimum hurdle rate of return?

The objective of this scenario is to show that the correct answer to the problem depends on whether correlation is ignored, as is often the case in practice, or incorporated into the problem. In credit risk analysis correlation is not only a problem it is more often than not **THE PROBLEM**.

Setting Up The Problem's Solution

We will define E_1 to be the default status of ABC Company. If ABC defaults on the loan then $E_1 = 1$. If ABC does not default on the loan then $E_0 = 1$. As noted in the problem above the unconditional probabilities associated with E_1 are...

$$P\left[E_1 = 1\right] = 0.20 \text{ ...and... } P\left[E_1 = 0\right] = 1 - 0.20 = 0.80 \tag{1}$$

We will define E_2 to be the default status of XYZ Company. If XYZ does not have the financial wherewithal to cover the guarantee if called upon to do so then $E_2 = 1$. If XYZ does have the financial wherewithal to cover the guarantee if called upon to do so then $E_2 = 0$. As noted in the problem above the unconditional probabilities associated with E_2 are...

$$P\left[E_2 = 1\right] = 0.10 \text{ ...and... } P\left[E_2 = 0\right] = 1 - 0.10 = 0.90 \tag{2}$$

The table below presents the bank's expected payoffs, which are a function of the default status of ABC Company, the default status of XYZ Company, the payoff given default status and the attendant probabilities...

Expected Payoff Table:

E_1	E_2	Payoff	Probability	Expected Payoff
0	0	1,100,000	?	?
1	0	1,100,000	?	?
0	1	1,100,000	?	?
1	1	300,000	?	?
Total			1.0000	?

All that is needed to finish the table above and calculate the expected payoff are the default probabilities.

We can redefine the probability that $E_1 = 1$ (ABC defaults on the loan) as the expected value of E_1. Using Equation (1) above the proof of this statement is...

$$\mathbb{E}\Big[E_1\Big] = 0 \times P\Big[E_1 = 0\Big] + 1 \times P\Big[E_1 = 1\Big] = P\Big[E_1 = 1\Big] \tag{3}$$

We can redefine the probability that $E_2 = 1$ (XYZ is not financially able to cover the guarantee if called upon to do so) as the expected value of E_2. Using Equation (2) above the proof of this statement is...

$$\mathbb{E}\Big[E_2\Big] = 0 \times P\Big[E_2 = 0\Big] + 1 \times P\Big[E_2 = 1\Big] = P\Big[E_2 = 1\Big] \tag{4}$$

We will define theta to be the default correlation between ABC Company and XYZ Company. The equation for default correlation is...

$$\theta = \frac{\mathbb{E}[E_1 E_2] - \mathbb{E}[E_1]\mathbb{E}[E_2]}{\sqrt{Var(E_1)Var(E_2)}} \tag{5}$$

We can rewrite Equation (5) above as...

$$\mathbb{E}[E_1 E_2] = \theta\sqrt{Var(E_1)Var(E_2)} + \mathbb{E}[E_1]\mathbb{E}[E_2] \tag{6}$$

The probability that ABC Company will not default on the loan and XYZ Company is financially able to cover the guarantee even though it is not called upon to do so is...

$$\begin{aligned} P\Big[E_1 = 0 \cap E_2 = 0\Big] &= \Big\{1 - \mathbb{E}\Big[E_1\Big]\Big\}\Big\{1 - \mathbb{E}\Big[E_2\Big]\Big\} \\ &= \mathbb{E}\Big[1 - E_1\Big]\mathbb{E}\Big[1 - E_2\Big] \\ &= \mathbb{E}\Big[(1 - E_1)(1 - E_2)\Big] \\ &= \mathbb{E}\Big[1 - E_1 - E_2 + E_1 E_2\Big] \\ &= 1 - \mathbb{E}\Big[E_1\Big] - \mathbb{E}\Big[E_2\Big] + \mathbb{E}\Big[E_1 E_2\Big] \end{aligned} \tag{7}$$

The probability that ABC Company will not default on the loan and XYZ Company is not financially able to cover the guarantee even though it is not called upon to do so is...

$$\begin{aligned} P\Big[E_1 = 0 \cap E_2 = 1\Big] &= \Big\{1 - \mathbb{E}\Big[E_1\Big]\Big\}\mathbb{E}\Big[E_2\Big] \\ &= \mathbb{E}\Big[1 - E_1\Big]\mathbb{E}\Big[E_2\Big] \\ &= \mathbb{E}\Big[(1 - E_1)E_2\Big] \\ &= \mathbb{E}\Big[E_2 - E_1 E_2\Big] \\ &= \mathbb{E}\Big[E_2\Big] - \mathbb{E}\Big[E_1 E_2\Big] \end{aligned} \tag{8}$$

The probability that ABC Company will default on the loan and XYZ Company will not default on the guarantee and payoff the loan per the terms of the guarantee is...

$$
\begin{aligned}
P\Big[E_1 = 1 \cap E_2 = 0\Big] &= \mathbb{E}\Big[E_1\Big]\Big\{1 - \mathbb{E}\Big[E_2\Big]\Big\} \\
&= \mathbb{E}\Big[E_1\Big]\mathbb{E}\Big[1 - E_2\Big] \\
&= \mathbb{E}\Big[E_1(1 - E_2)\Big] \\
&= \mathbb{E}\Big[E_1 - E_1 E_2\Big] \\
&= \mathbb{E}\Big[E_1\Big] - \mathbb{E}\Big[E_1 E_2\Big]
\end{aligned} \tag{9}
$$

The probability that ABC Company will default on the loan and XYZ Company will default on the guarantee is...

$$
\begin{aligned}
P\Big[E_1 = 1 \cap E_2 = 1\Big] &= \mathbb{E}\Big[E_1\Big]\mathbb{E}\Big[E_2\Big] \\
&= \mathbb{E}\Big[E_1 E_2\Big]
\end{aligned} \tag{10}
$$

The Problem Solution Assuming No Correlation

Using Equation (1) the expected value of $E1$ is...

$$\mathbb{E}\Big[E_1\Big] = P\Big[E_1 = 1\Big] = 0.20 \tag{11}$$

Using Equation (2) the expected value of $E2$ is...

$$\mathbb{E}\Big[E_2\Big] = P\Big[E_2 = 1\Big] = 0.10 \tag{12}$$

Using a default correlation of zero and Equations (1), (2), (6) and (21) the expected value of the product of $E1$ and $E2$ is...

$$\mathbb{E}[E_1 E_2] = \theta\sqrt{Var(E_1)Var(E_2)} + \mathbb{E}[E_1]\mathbb{E}[E_2] = (0)(0.12) + (0.20)(0.10) = 0.0200 \tag{13}$$

The expected payoff to the bank is...

E_1	E_2	Payoff	Probability Calculation	Probability	Expected Payoff	Reference
0	0	1,100,000	$1 - \mathbb{E}[E_1] - \mathbb{E}[E_2] + \mathbb{E}[E_1 E_2]$	0.7200	792,000	Equation (7)
1	0	1,100,000	$\mathbb{E}[E_2] - \mathbb{E}[E_1 E_2]$	0.0800	88,000	Equation (8)
0	1	1,100,000	$\mathbb{E}[E_1] - \mathbb{E}[E_1 E_2]$	0.1800	198,000	Equation (9)
1	1	300,000	$\mathbb{E}[E_1 E_2]$	0.0200	6,000	Equation (10)
Total				1.0000	1,084,000	

The Problem's Answer: The bank's total expected return on the proposed transaction is \$1,084,000 or 8.40%, which is greater than the bank's minimum hurdle rate of return of 5.00%. The loan interest rate of 10% is high enough to support the bank's minimum hurdle rate of return.

The Problem Solution Assuming Correlation

Using Equation (1) the expected value of $E1$ is...

$$\mathbb{E}\Big[E_1\Big] = P\Big[E_1 = 1\Big] = 0.20 \tag{14}$$

Using Equation (2) the expected value of $E2$ is...

$$\mathbb{E}\bigg[E_2\bigg] = P\bigg[E_2 = 1\bigg] = 0.10 \tag{15}$$

Using a default correlation of 0.60 and Equations (1), (2), (6) and (21) the expected value of the product of $E1$ and $E2$ is...

$$\mathbb{E}[E_1 E_2] = \theta\sqrt{Var(E_1)Var(E_2)} + \mathbb{E}[E_1]\mathbb{E}[E_2] = (0.60)(0.12) + (0.20)(0.10) = 0.0920 \tag{16}$$

The expected payoff to the bank is...

E_1	E_2	Payoff	Probability Calculation	Probability	Expected Payoff	Reference
0	0	1,100,000	$1 - \mathbb{E}[E_1] - \mathbb{E}[E_2] + \mathbb{E}[E_1 E_2]$	0.7920	871,200	Equation (7)
1	0	1,100,000	$\mathbb{E}[E_2] - \mathbb{E}[E_1 E_2]$	0.0080	8,800	Equation (8)
0	1	1,100,000	$\mathbb{E}[E_1] - \mathbb{E}[E_1 E_2]$	0.1080	118,800	Equation (9)
1	1	300,000	$\mathbb{E}[E_1 E_2]$	0.0920	27,600	Equation (10)
Total				1.0000	1,026,400	

The Problem's Answer: The bank's total expected return on the proposed transaction is $1,026,400 or 2.64%, which is less than the bank's minimum hurdle rate of return of 5.00%. The loan interest rate of 10% is not high enough to support the bank's minimum hurdle rate of return.

Conclusion

By not modeling correlation the Credit Analyst would come to the incorrect conclusion that the interest rate on the loan results in a rate of return to the bank that was greater the the bank's minimum hurdle rate of return. The probability that both ABC Company will default on the loan and XYZ Company will default on the guarantee is 2.0% if correlation is incorrectly ignored and 9.2% if correlation is properly included in the analysis.

Appendix

A. The expected value of the square of E_1 is...

$$\begin{aligned}\mathbb{E}\bigg[E_1^2\bigg] &= 0^2 \times P\bigg[E_1 = 0\bigg] + 1^2 \times P\bigg[E_1 = 1\bigg] \\ &= P\bigg[E_1 = 1\bigg]\end{aligned} \tag{17}$$

B. The expected value of the square of E_2 is...

$$\begin{aligned}\mathbb{E}\bigg[E_2^2\bigg] &= 0^2 \times P\bigg[E_2 = 0\bigg] + 1^2 \times P\bigg[E_2 = 1\bigg] \\ &= P\bigg[E_2 = 1\bigg]\end{aligned} \tag{18}$$

C. The variance of E_1 is...

$$\begin{aligned}Var(E_1) &= \mathbb{E}\bigg[E_1^2\bigg] - \bigg(\mathbb{E}\bigg[E_1\bigg]\bigg)^2 \\ &= P\bigg[E_1 = 1\bigg] - \bigg(P\bigg[E_1 = 1\bigg]\bigg)^2 \\ &= P\bigg[E_1 = 1\bigg]\bigg(1 - P\bigg[E_1 = 1\bigg]\bigg) \\ &= (0.20)(1 - 0.20) \\ &= 0.1600\end{aligned} \tag{19}$$

D. The variance of E_2 is...

$$\begin{aligned}
Var(E_2) &= \mathbb{E}\Big[E_2^2\Big] - \Big(\mathbb{E}\Big[E_2\Big]\Big)^2 \\
&= P\Big[E_2 = 1\Big] - \Big(P\Big[E_2 = 1\Big]\Big)^2 \\
&= P\Big[E_2 = 1\Big]\Big(1 - P\Big[E_2 = 1\Big]\Big) \\
&= (0.10)(1 - 0.10) \\
&= 0.0900 \qquad (20)
\end{aligned}$$

E. Using Equations (19) and (20) the square root of the product of the variances of E_1 and E_2 is...

$$\sqrt{Var(E_1)Var(E_2)} = \sqrt{(0.1600)(0.0900)} = 0.1200 \qquad (21)$$